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A Note on Computing the Crisp Order Context of a Fuzzy Formal Context for Knowledge Reduction

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Abstract

Fuzzy Formal Concept Analysis (FCA) is a mathematical tool for the effective representation of imprecise and vague knowledge. However, with a large number of formal concepts from a fuzzy context, the task of knowledge representation becomes complex. Hence, knowledge reduction is an important issue in FCA with a fuzzy setting. The purpose of this current study is to address this issue by proposing a method that computes the corresponding crisp order for the fuzzy relation in a given fuzzy formal context. The obtained formal context using the proposed method provides a fewer number of concepts when compared to original fuzzy context. The resultant lattice structure is a reduced form of its corresponding fuzzy concept lattice and preserves the specialized and generalized concepts, as well as stability. This study also shows a step-by-step demonstration of the proposed method and its application.

Keywords

Crisp Context, Concept Lattice, Formal Concept Analysis, Fuzzy Formal Concept, Fuzzy Relation, Knowledge Reduction

1. Introduction

In the early 1980s, Wille [1] proposed a mathematical model, called the Formal Concept Analysis (FCA), for conceptual data analysis and knowledge processing tasks. This theory is associated with a formal context (**G**, **M**, *I*) in which G represents a set of formal objects, **M** represents a set of formal attributes, and I is the binary relation between them. The main outputs of FCA are formal concepts, concept lattices, and implications from a given formal context [2]. A formal concept represents a set of objects, which are called the extent, and its common attributes, which are called the intent. All of which are closed with the Galois connection. The concept lattice represents a hierarchical order among the generated formal concepts in the form of specialization and generalization. FCA has been successfully applied for data mining, information retrieval, and knowledge discovery tasks in various fields, as discussed by Carpineto and Romano [3]. Burusco and Fuentes-Gonzalez [4] incorporated FCA with a fuzzy setting for handling uncertainty and imprecision. After that, several approaches were proposed for generating the fuzzy concept lattice [5]. FCA with a fuzzy setting has been successfully applied in different applications including mathematical searches, information retrieval, and association rule

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mining [5-9]. In this process a major problem is the size of the concept lattice constructed from a large context. Hence, knowledge reduction is an important issue in FCA [2-19]. Knowledge reduction discusses reducing the size of the concept lattice, attributes (objects), and the number of formal concepts to avoid redundancy while maintaining the structure consistency. For this purpose, several approaches have been proposed, which we will discuss in Section 2 extensively. In this paper, we focused on computing the crisp order relation of a given fuzzy relation in the fuzzy formal context to encounter the issue [20-22]. The aim of this current study is to reduce the number of fuzzy formal concepts and its lattice structure. The proposed method provides a corresponding crisp formal context for a given fuzzy formal context in the following cases:

Case (1) Number of objects (O_i) $\neq$ Number of attributes (A_j).

Case (2) Number of objects (O_i) = Number of attributes (A_j).

Our study also shows a step-by-step demonstration of the corresponding crisp order relation of the given fuzzy formal context. For the purpose of validation we have used the following metrics: (a) the availability of generalized and specialized concepts generated from the fuzzy formal context and its corresponding crisp order context, and (b) the stability of the obtained formal concepts using the proposed method. We applied the proposed method on a fuzzy data set discussed by Kandasamy and Smarandache [23].

The rest of the paper is organized as follows: Section 2 provides a brief background about FCA in the fuzzy setting. In Section 3 we introduce the proposed method. In Section 4 we provide illustrations of the proposed method. Section 5 demonstrates an application of the proposed method. Section 6 contains discussions, followed by a presentation of the conclusion, acknowledgements, and references.

2. Formal Concept Analysis in the Fuzzy Setting

A fuzzy formal context is a triplet **K** = (**G**, **M**, $\underset{\sim}{R}$) where **G** is set of formal objects, **M** is a set of formal attributes, and $\underset{\sim}{R}$ is a fuzzy relation between **G** and **M** [4,5]. The fuzzy relation $\underset{\sim}{R}(g,m)$ represents that the object $g \in \mathbf{G}$ has a membership value $\underset{\sim}{\mu}(g,m)$ with the attributes $m \in \mathbf{M}$. There are different possibilities for a formal context in FCA based on the type of objects, attributes, and the fuzzy relation.

Table 1. Some possible conditions in a given fuzzy formal context

Condition	Object	Attribute	Fuzzy relation
(a)	Complete	Complete	Incomplete
(b)	Incomplete	Complete	Complete
(c)	Complete	Incomplete	Complete
(d)	Incomplete	Incomplete	Complete
(e)	Crisp	Crisp	Complete
(f)	Crisp	Fuzzy	Complete
(g)	Fuzzy	Crisp	Complete
(h)	Fuzzy	Fuzzy	Complete

Table 1 provides some possible conditions in a given fuzzy formal context. Very recently, a few investigations have been available in the FCA literature for an incomplete fuzzy relation, condition (a), as shown by in Table 1 [24-27].

In this study we restricted our analysis to the possibilities of when the fuzzy relation is complete, (i.e., conditions (b)–(h)). The notions of the residuated lattice, fuzzy Galois connection, fuzzy closure property, and complete lattice are defined in brief below.

A fuzzy set with a binary relation $\leq$ on a set $\underset{\sim}{S}$ is called the partial order relation iff [28]:

- Reflexive: $\underset{\sim}{x} \leq \underset{\sim}{x}$, $\forall$ $\underset{\sim}{x} \in \underset{\sim}{S}$,
- Anti symmetric: $\underset{\sim}{x} \leq \underset{\sim}{y}$ and $\underset{\sim}{y} \leq \underset{\sim}{x} \rightarrow \underset{\sim}{x} \leq \underset{\sim}{z}$, $\forall$ $\underset{\sim}{x}, \underset{\sim}{y} \in \underset{\sim}{S}$,
- Transitive: $\underset{\sim}{x} \leq \underset{\sim}{y}$ and $\underset{\sim}{y} \leq \underset{\sim}{z} \rightarrow \underset{\sim}{x} \leq \underset{\sim}{z}$, $\forall$ $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z} \in \underset{\sim}{S}$.

A fuzzy lattice is a partially ordered set of ($\underset{\sim}{S}$, ≤), in which for every pair of ($\underset{\sim}{x}, \underset{\sim}{y}$), there exists a Supremum $= \underset{\sim}{x} \vee \underset{\sim}{y}$ and an Infimum $= \underset{\sim}{x} \wedge \underset{\sim}{y}$. The residuated lattice $\mathbf{L} = (\mathrm{L}, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is the finite structure of truth-values of the object and its properties. **L** is complete residuated lattice iff [5, 29-30]:

(1) $(\mathrm{L}, \wedge, \vee, 0, 1)$ is a complete lattice.

(2) $(\mathrm{L}, \otimes, 1)$ is commutative monoid. (i.e., $\otimes$ is the commutative and associative means $a \otimes 1 = 1 \otimes a = a$, $\forall$ $a \in$ **L**).

(3) $\otimes$ and $\rightarrow$ are the binary operations are called multiplication and residuum, respectively.

The operators $\otimes$ and $\rightarrow$ are defined distinctly by Lukasiewicz, Gödel, and Goguen [7].

Lukasiewicz: $a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$.

Godel: $a \otimes b = \min(a,\ b)$ $a \rightarrow b = 1$ if $a \leq b$ otherwise b.

Goguen: $a \otimes b = a.b$, $a \rightarrow b = 1$ if $a \leq b$ otherwise b/a .

For any L-set of $O \in \mathrm{L}^{\mathrm{G}}$ objects and L- set of $A \in \mathrm{L}^{\mathrm{M}}$ attributes, we can define an L-set of $O^{\uparrow} \in \mathrm{L}^{\mathrm{G}}$ attributes and an L-set of $A^{\downarrow} \in \mathrm{L}^{\mathrm{M}}$ objects, respectively, as follows [29,30]:

- $O^{\uparrow}(m) = \wedge_{g \in G}(O(g) \rightarrow \underset{\sim}{R}(g,m))$
- $A^{\downarrow}(g) = \wedge_{m \in M}(A(m) \rightarrow \underset{\sim}{R}(g,m))$

The $O^{\uparrow}(m)$ is the truth degree of the attribute m is covered by all objects from g and $A^{\downarrow}(g)$ is the truth degree of object g that has all the attributes from m. The fuzzy formal concept is a pair of $(O, A) \in \mathrm{L}^{\mathrm{G}} \times \mathrm{L}^{\mathrm{M}}$, such that $O^{\uparrow} = A$ and $A^{\downarrow} = O$, where the fuzzy set of objects O are called the extent and the fuzzy set of attributes A that are called intents. The operator $(\uparrow, \downarrow)$ is known as a fuzzy Galois connection for extensive study readers can refer to [24,25,29-35]. When the operator ($\uparrow$) is applied to a set of objects, it provides a set of attributes that are covered by these objects. Consequently, when the operator ($\downarrow$) is applied to these covered attributes, we can find the additional objects that may cover these attributes. Hence, the fuzzy formal concept is a maximal rectangle of a given fuzzy formal context **K** filled with a membership value between [0, 1], which is an ordered pair of two sets (O, A), where $O \subseteq$ **G** is called the extent, and $A \subseteq$ **M** is called the intent iff they form the fuzzy closure property, which is as defined below.

Two fuzzy closure operators can be defined as, ϕ ($\uparrow\downarrow$): $L^G \to L^G$ and ψ ($\downarrow\uparrow$): $L^M \to L^M$, $\forall$ O_1, O_2, $O \in L^G$ and A_1, A_2, $A \in L^M$ satisfy following properties [31,35]:

- $O_1 \subseteq O_2 \rightarrow \phi(O_1) \subseteq \phi(O_2)$ and $A_1 \subseteq A_2 \rightarrow \psi(A_1) \subseteq \psi(A_2)$
- $O \subseteq \phi(O)$ and $A \subseteq \psi(A)$,
- $\phi(\phi(O)) = \phi(O)$ and $\psi(\psi(A)) = \psi(A)$

Through these closure properties one can neither enlarge the attributes nor the objects of a fuzzy formal concept. The set of fuzzy formal concepts FC_K follows the super and sub hierarchy properties $(O_2, A_2) \leq (O_1, A_1) :\Leftrightarrow O_2 \subseteq O_1$ ($\Leftrightarrow$ $A_2 \supseteq A_1$) in the lattice structure $L_{FC_K} = (FC_K, \leq)$. Together with this ordering, in the complete lattice an infimum (0, 0, …, 0) and a supremum (1,1,…,1) exist for some formal concepts [30,31]:

- $\wedge_{j\in J}(O_j, A_j) = (\bigcap_{j\in J} O_j, (\bigcup_{j\in J} A_j)^{\downarrow\uparrow})$
- $\vee_{j\in J}(O_j, A_j) = ((\bigcup_{j\in J} O_j)^{\uparrow\downarrow}, \bigcap_{j\in J} A_j)$

Table 2. Summary of some important references on the knowledge reduction issue

FCA in crisp and fuzzy settings	FCA through granular and threshold	Decomposition methods in FCA	Approximation methods in FCA and its extension	Reduction methods in FCA	Extensive study of fuzzy FCA
Carpineto and Romano [3]	Hu et al. [14]	Gely [12]	Nguyen et al. [9]	Ganter and Wille [2]	Ayouni et al. [6]
Burusco and Gonzalez [4]	Prem Kumar and Aswani Kumar [18]	Guo et al. [13]	Hu et al. [14]	Carpineto and Romano [3]	Maio et al. [8]
Belohlavek and Vychodil [5]	Belohlavek [29,30]	Hu et al. [14]	Djouadi [24]	Ghosh et al. [7]	Kandasamy and Smarandache [23]
Aswani Kumar [10,11,45,46,53,54]	Zhang et al. [32]	Li and Zhang [16]	Dubois and Prade [25]	Aswani Kumar and Srinivas [10]	Djouadi [24], Dubois and Prade [25]
Beg and Ashraf [20]	Cross and Kandasamy [36]	Prem Kumar and Aswani Kumar [19]	Krupka and Lastovica [26]	Konecny and Krupka [15]	Prem Kumar and Aswani Kumar [51,52]
Lee [21], Zadeh [22]	Wu et al. [39]	Beg and Ashraf [20]	Li et al. [27,]	Li and Zhang [16]	Belohlavek and Konecny [33]
Kuznetsov and Obiedkov [37,40]	Kang et al. [38]	Dubois and Prade [25]	Pocs [31]	Li et al. [17,48,49]	Medina [34]
Yang et al. [51]	Liu et al. [43]	Horvath et al. [42]	Shao et al. [35], Skowron et al. [44]	Bartl et al. [41]	Skowron et al. [44]

For detailed illustrations about generating the formal concepts from a given formal context, readers can refer to references including [1-8,13,16,24-27,29-40]. Reducing the number of formal concepts and the size of the lattice structure are open issues for researchers as knowledge reduction problems. Table 2 summarizes the approaches that are available to handle these issues. Our proposed method computes the corresponding crisp order for the fuzzy relation in the given fuzzy formal context for reducing the number of fuzzy formal concepts and the size of the lattice structure from a fuzzy context.

3. Proposed Method

In this section, we propose a method for computing the crisp order relation for the fuzzy relation of a fuzzy context for the two cases mentioned in Section 1.

If $\underset{\sim}{R}_{ij}$ is a fuzzy relation, then it can be transformed into a corresponding crisp relation R'_{ij} as defined below:

Step 1. If $\mu_R(x,y) \geq \mu_R(y,x)$, then $\mu_{R'}(x,y)=1, \mu_{R'}(y,x)=0$.

Step 2. If $\mu_R(x,y) = \mu_R(y,x)$, then $\mu_{R'}(x,y)=\mu_{R'}(y,x)$ where

$\mu_{R'}(x,y)$ =1 if $\mu_R(x, y) = 1$ otherwise 0.

Step 3. For other conditions $\underset{\sim}{R}'_{ij}$ = 1 if $\underset{\sim}{R}_{ij} \geq \delta$ otherwise 0.

The pair (x, y) is considered as the object and attribute, respectively, in the fuzzy formal context. These pairs can be visualized as a hierarchical order in the concept lattice so that they can be compared or ordered. In the proposed method, some possibilities for a given fuzzy formal context are also considered as described below:

Case 1. Number of objects (O_i) $\neq$ Number of attributes (A_j).

Step 1. If $\mu_R(O_i,A_j) \geq \mu_R(O_j,A_i)$ then $\mu_{R'}(O_i,A_j)=1, \mu_{R'}(O_j,A_i)=0$.

Similarly when $\mu_R(O_i,A_j) \leq \mu_R(O_j,A_i)$ then $\mu_{R'}(O_i,A_j)=0, \mu_{R'}(O_j,A_i)=1$.

Step 2. If $\mu_R(O_i,A_j) = \mu_R(O_j,A_i)$ then $\mu_{R'}(O_i,A_j)=\mu_{R'}(O_j,A_i)$

where , $\mu_{R'}(O_i,A_j)=1$, If $\mu_R(O_i,A_j)=1$ otherwise 0.

Step 3. If $\mu_R(O_i,A_i)>0$ then $\mu_{R'}(O_i,A_i)=1$. In other conditions if, $\mu_R(O_i,A_j)>0$ then $\mu_{R'}(O_i,A_j)=1$.

Case 2. Number of objects (O_i) = Number of attributes (A_j).

Step 1. If $\mu_R(O_i,A_j) \geq \mu_R(O_j,A_i)$ then $\mu_{R'}(O_i,A_j)=1, \mu_{R'}(O_j,A_i)=0$.

Similarly, when $\mu_{R'}(O_i,A_j)=1, \mu_{R'}(O_j,A_i)=0$ then $\mu_{R'}(O_i,A_j)=0, \mu_{R'}(O_j,A_i)=1$.

Step 2. If $\mu_R(O_i,A_j) = \mu_R(O_j,A_i)$ then $\mu_{R'}(O_i,A_j)=\mu_{R'}(O_j,A_i)$

where , $\mu_{R'}(O_i,A_j)=\mu_{R'}(O_j,A_i)$ = If $\mu_R(O_i,A_j)=1$ then 1 otherwise 0.

Step 3. In other conditions: if $\mu_R(O_i,A_i)>0$ then $\mu_{R'}(O_i,A_i)=1$.

We can observe that the proposed method computes the corresponding crisp order relation of the given fuzzy relation between the objects and the attributes. The maximum number of fuzzy relations in any given fuzzy formal context cannot exceed (**|G|** × **|M|**). Hence, the proposed method takes the maximum $2\times$ (**|G|** × **|M|**) complexity for computing the crisp order relation. The complexity for building the crisp concept lattice is usually O (**|G|**+**|M|**).**|M|**.**|L|**), as discussed by Kuznetsov and Obiedkov [37]. The overall complexity of the proposed method to compute the crisp order context and to construct its lattice structure is O (**|G|**× **|M|** + (**|G|**+**|M|**).**|M|**.**|L|**)).

4. Illustrations

4.1 Illustration of the Proposed Method

To illustrate the proposed method we have considered two fuzzy formal contexts, which are shown in Tables 3 and 5. Table 3 represents a fuzzy formal context in which the number of objects (O_i) $\neq$ number of attributes (A_j) (Case 1) [8]. Table 4 represents the corresponding crisp context of Table 3 using the proposed method. Table 5 represents a fuzzy formal context in which the number of objects (O_i) = number of attributes (A_j) (Case 2) [7]. Table 6 represents the corresponding crisp context of Table 5 using the proposed method.

Case 1 illustration of the proposed method:

The fuzzy formal concepts generated from the fuzzy context shown in Table 3 are:

Table 3. A fuzzy context in which the number of objects $\neq$ the number of attributes

	A_1	A_2	A_3	A_4
O_1	0.61	0.89	0.00	0.00
O_2	0.94	0.00	0.71	1.00
O_3	1.00	0.00	0.00	0.76
O_4	0.70	0.00	0.97	0.00
O_5	0.78	0.64	1.00	1.00

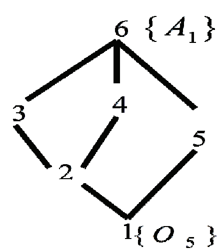

Fig. 1. Fuzzy concept lattice for the context shown in Table 3.

1. $\{\{0.64/O_5\},\{1.0/A_1+1.0/A_2+1.0/A_3+1.0/A_4\}\}$
2. $\{\{0.71/O_2+0.78/O_5\},\{1.0/A_1+1.0/A_3+1.0/A_4\}\}$
3. $\{\{0.94/O_2+0.76/O_3+0.78/O_5\},\{1.0/A_1+1.0/A_4\}\}$
4. $\{\{0.71/O_2+0.70/O_4+0.78/O_5\},\{1.0/A_1+1.0/A_3\}\}$
5. $\{\{0.61/O_1+0.64/O_5\},\{1.0/A_1+1.0/A_2\}\}$
6. $\{\{0.61/O_1+0.94/O_2+1.00/O_3+0.70/O_4+0.78/O_5\},\{1.0/A_1\}\}$

The line diagram of concepts generated from the fuzzy context of Table 3 is shown in Fig. 1 [8].

From the fuzzy concept lattice shown in Fig. 1 we can conclude that:

- A_1 is a generalized attribute, which covers the maximal objects of the fuzzy formal context, as shown in Table 3,
- O_5 is a specialized object, which covers the maximal attributes of the fuzzy formal context, as

shown in Table 3.

The computed crisp order relations for the fuzzy relation shown in Table 3 (using the proposed method) are:

1. $\mu_R(O_1,A_1)\ge 0$. Hence, $\mu_{R'}(O_1,A_1)=1$.
2. $\mu_R(O_1,A_2)\le \mu_R(O_2,A_1)$. Hence, $\mu_{R'}(O_1,A_2)=0, \mu_{R'}(O_2,A_1)=1$.
3. $\mu_R(O_1,A_3)\le \mu_R(O_3,A_1)$. Hence, $\mu_{R'}(O_1,A_3)=0, \mu_{R'}(O_3,A_1)=1$.
4. $\mu_R(O_1,A_4)\le \mu_R(O_4,A_1)$. Hence, $\mu_{R'}(O_1,A_4)=0, \mu_{R'}(O_4,A_1)=1$.
5. $\mu_R(O_2,A_2)=0$. Hence, $\mu_{R'}(O_2,A_2)=0$.
6. $\mu_R(O_2,A_3)\ge \mu_R(O_3,A_2)$. Hence, $\mu_{R'}(O_2,A_3)=1, \mu_{R'}(O_3,A_2)=0$.
7. $\mu_R(O_2,A_4)\ge \mu_R(O_4,A_2)$. Hence, $\mu_{R'}(O_2,A_4)=1, \mu_{R'}(O_4,A_2)=0$.
8. $\mu_R(O_3,A_3)=0$. Hence, $\mu_{R'}(O_3,A_3)=0$.
9. $\mu_R(O_3,A_4)\le \mu_R(O_4,A_3)$. Hence, $\mu_{R'}(O_3,A_4)=0, \mu_{R'}(O_4,A_3)=1$.
10. $\mu_R(O_4,A_4)=0$. Hence, $\mu_{R'}(O_4,A_4)=0$.
11. $\mu_R(O_5,A_1)\ge 0$. Hence, $\mu_{R'}(O_5,A_1)=1$.

Similarly, $\mu_{R'}(O_5,A_2)=1, \mu_{R'}(O_5,A_3)=1, \mu_{R'}(O_5,A_4)=1$.

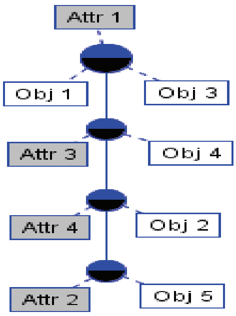

Fig. 2. Concept lattice for the context shown in Table 4.

The computed crisp order relations for the fuzzy relation shown in Table 3 are tabulated as the crisp context in Table 4.

Table 4. Crisp order for the fuzzy context of Table 3

	A_1	A_2	A_3	A_4
O_1	1	0	0	0
O_2	1	0	1	1
O_3	1	0	0	0
O_4	1	0	1	0
O_5	1	1	1	1

The formal concepts generated from the context shown in Table 4 are:

1. $\{\{O_5\},\{A_1,A_2,A_3,A_4\}\}$,
2. $\{\{O_2,O_5\},\{A_1,A_3,A_4\}\}$,

3. $\{\{O_2, O_4, O_5\}, \{A_1, A_3\}\}$,
4. $\{\{O_1, O_2, O_3, O_4, O_5\}, \{A_1\}\}$.

The line diagram of concepts generated from the formal context of Table 4 is shown in Fig. 2, from which we can conclude that:

- A_1 is a generalized attribute, which covers the maximal objects of the formal context, as shown in Table 4,
- O_5 is a specialized object, which covers the maximal attributes of the formal context, as shown in Table 4.

We can observe that the fuzzy concept lattice shown in Fig. 1 and its corresponding crisp lattice shown in Fig. 2 have the same specialized and generalized concepts. We can also observe that the crisp lattice structure (Fig. 2) contains a fewer number of concepts when compared to the corresponding fuzzy concept lattice (Fig. 1).

Table 5. A fuzzy formal context in which number of objects=number of attributes

	A_1	A_2	A_3	A_4	A_5	A_6
O_1	0	1	0.5	0.5	1	0
O_2	1	1	1	0	0	0
O_3	0.5	0.5	0	0	0	1
O_4	0	0	0	1	0.5	0
O_5	0	0	1	0.5	0	0
O_6	0.5	0	0	0	0	0

Case 2 illustration of the proposed method:

The fuzzy formal concepts obtained from the fuzzy context shown in Table 5 are:

1. $\{\{\varnothing\}, \{1.0/A_1 + 1.0/A_2 + 1.0/A_3 + 1.0/A_4 + 1.0/A_5 + 1.0/A_6\}\}$
2. $\{\{0.5/O_1\}, \{1.0/A_2 + 1.0/A_3 + 1.0/A_4 + 1.0/A_5\}\}$
3. $\{\{O_2\}, \{1.0/A_1 + 1.0/A_2 + 1.0/A_3\}\}$
4. $\{\{0.5/O_3\}, \{1./0/A_1 + 1.0/A_2 + 1.0/A_6\}\}$
5. $\{\{0.5/O_1 + 0.5/O_5\}, \{1.0/A_3 + 1.0/A_4\}\}$
6. $\{\{0.5/O_1 + 0.5/O_4\}, \{1.0/A_4 + 1.0/A_5\}\}$
7. $\{\{O_1\}, \{1.0/A_2 + 0.5/A_3 + 0.5/A_4 + 1.0/A_5\}\}$
8. $\{\{0.5/O_1 + 1.0/O_2\}, \{1.0/A_2 + 1.0/A_3\}\}$
9. $\{\{1.0/O_2 + 0.5/O_3\}, \{1.0/A_1 + 1.0/A_2\}\}$
10. $\{\{O_3\}, \{0.5/A_1 + 0.5/A_2 + 1.0/A_6\}\}$
11. $\{0.5/O_1 + 1.0/O_5\}, \{1.0/A_3 + 0.5/A_4\}\}$
12. $\{\{0.5/O_1 + 1.0/O_4\}, \{1.0/A_4 + 0.5/A_5\}\}$
13. $\{\{1.0/O_1 + 0.5/O_4\}, \{0.5/A_4 + 1.0/A_5\}\}$
14. $\{\{1.0/O_1 + 1.0/O_5\}, \{0.5/A_3 + 0.5/A_4\}\}$
15. $\{\{0.5/O_1 + 1.0/O_2 + 1.0/O_5\}, \{1.0/A_3\}\}$

16. $\{\{1.0/O_1 + 1.0/O_2\}, \{1.0/A_2 + 0.5/A_3\}\}$
17. $\{\{1.0/O_2 + 0.5/O_3 + 0.5/O_6\}, \{1.0/A_1\}\}$
18. $\{\{1.0/O_2 + 1.0/O_3\}, \{0.5/A_1 + 0.5/A_2\}\}$
19. $\{\{0.5/O_1 + 1.0/O_4 + 0.5/O_5\}, \{1.0/A_4\}\}$
20. $\{\{1.0/O_1 + 1.0/O_4\}, \{0.5/A_4 + 0.5/A_5\}\}$
21. $\{\{1.0/O_1 + 1.0/O_2 + 1.0/O_5\}, \{0.5/A_3\}\}$
22. $\{\{1.0/O_1 + 1.0/O_2 + 0.5/O_3\}, \{1.0/A_2\}\}$
23. $\{\{1.0/O_1 + 1.0/O_4 + 1.0/O_5\}, \{0.5/A_4\}\}$
24. $\{\{1.0/O_1 + 1.0/O_2 + 1.0/O_3\}, \{0.5/A_2\}\}$
25. $\{\{1.0/O_2 + 1.0/O_3 + 1.0/O_6\}, \{0.5/A_1\}\}$
26. $\{\{1.0/O_1 + 1.0/O_2 + 1.0/O_3 + 1.0/O_4 + 1.0/O_5 + 1.0/O_6\}, \varnothing\}$

Where $\varnothing$ represents a null set.

The line diagram (fuzzy concept lattice) of these fuzzy concepts is shown in Fig. 3, from which we can conclude that:

- The specialized concepts in Fig. 3 are 2, 3, and 4. Hence, objects O_1, O_2, and O_3 cover the maximal attributes of the fuzzy formal context, as shown in Table 5.
- The generalized concepts are 21, 23, 24, and 25. Hence, attributes A_1, A_2, A_3 and A_4 cover the maximal objects of the fuzzy formal context, as shown in Table 5.

The corresponding crisp order relations for fuzzy relations that are shown in Table 5 are:

1. $\mu_R(O_1, A_1) = 0$. Hence, $\mu_{R'}(O_1, A_1) = 0$.
2. $\mu_R(O_1, A_2) = \mu_R(O_2, A_1) = 1$. Hence, $\mu_{R'}(O_1, A_2) = \mu_{R'}(O_2, A_1) = 1$.
3. $\mu_R(O_1, A_3) = \mu_R(O_3, A_1)$. Hence, $\mu_{R'}(O_1, A_3) = 0, \mu_{R'}(O_3, A_1) = 0$.
4. $\mu_R(O_1, A_4) \geq \mu_R(O_4, A_1)$. Hence, $\mu_{R'}(O_1, A_4) = 1, \mu_{R'}(O_4, A_1) = 0$.
5. $\mu_R(O_1, A_5) \geq \mu_R(O_5, A_1)$. Hence, $\mu_{R'}(O_1, A_5) = 1, \mu_{R'}(O_5, A_1) = 0$.
6. $\mu_R(O_1, A_6) \leq \mu_R(O_6, A_1)$. Hence, $\mu_{R'}(O_1, A_6) = 0, \mu_{R'}(O_6, A_1) = 1$.
7. $\mu_R(O_2, A_2) \geq 0$. Hence, $\mu_{R'}(O_2, A_2) = 1$.
8. $\mu_R(O_2, A_3) \geq \mu_R(O_3, A_2)$. Hence, $\mu_{R'}(O_2, A_3) = 1, \mu_{R'}(O_3, A_2) = 0$.
9. $\mu_R(O_2, A_4) = \mu_R(O_4, A_2)$. Hence, $\mu_{R'}(O_2, A_4) = 0, \mu_{R'}(O_4, A_2) = 0$.
10. $\mu_R(O_2, A_5) = \mu_R(O_5, A_2)$. Hence, $\mu_{R'}(O_2, A_5) = 0, \mu_{R'}(O_5, A_2) = 0$.
11. $\mu_R(O_2, A_6) = \mu_R(O_6, A_2)$. Hence, $\mu_{R'}(O_2, A_6) = 0, \mu_{R'}(O_6, A_2) = 0$.
12. $\mu_R(O_3, A_3) = 0$. Hence, $\mu_{R'}(O_3, A_3) = 0$.
13. $\mu_R(O_3, A_4) = \mu_R(O_4, A_3)$. Hence, $\mu_{R'}(O_3, A_4) = 0, \mu_{R'}(O_4, A_3) = 0$.
14. $\mu_R(O_3, A_5) \leq \mu_R(O_5, A_3)$. Hence, $\mu_{R'}(O_3, A_5) = 0, \mu_{R'}(O_5, A_3) = 1$.
15. $\mu_R(O_3, A_6) \geq \mu_R(O_6, A_3)$. Hence, $\mu_{R'}(O_3, A_6) = 1, \mu_{R'}(O_6, A_3) = 0$.
16. $\mu_R(O_4, A_4) = 1$. Hence, $\mu_{R'}(O_4, A_4) = 1$.

Table 6. Crisp order of the fuzzy context shown in Table 5

	A_1	A_2	A_3	A_4	A_5	A_6
O_1	0	1	0	1	1	0
O_2	1	1	1	0	0	0
O_3	0	0	0	0	0	1
O_4	0	0	0	1	0	0
O_5	0	0	1	0	0	0
O_6	1	0	0	0	0	0

17. $\mu_R(O_4, A_5) = \mu_R(O_5, A_4)$. Hence, $\mu_{R^{'}}(O_4, A_5) = 0, \mu_{R^{'}}(O_5, A_4) = 0$.
18. $\mu_R(O_4, A_6) = \mu_R(O_6, A_4)$. Hence, $\mu_{R^{'}}(O_4, A_6) = \mu_{R^{'}}(O_6, A_4) = 0$.
19. $\mu_R(O_5, A_5) = 0$. Hence, $\mu_{R^{'}}(O_5, A_5) = 0$.
20. $\mu_R(O_5, A_6) = \mu_R(O_6, A_5)$. Hence, $\mu_{R^{'}}(O_5, A_6) = \mu_{R^{'}}(O_6, A_5) = 0$.
21. $\mu_R(O_6, A_6) = 0$. Hence, $\mu_{R^{'}}(O_6, A_6) = 0$.

These relations are shown in Table 6, which represents the corresponding crisp order formal context of Table 5.

The formal concepts generated from the context shown in Table 6 are:

1. $\{\{\varnothing\}, \{A_1, A_2, A_3, A_4, A_5, A_6\}\}$
2. $\{\{O_3\}, \{A_6\}\}$
3. $\{\{O_1\}, \{A_2, A_4, A_5\}\}$
4. $\{\{O_2\}, \{A_1, A_2, A_3\}\}$
5. $\{\{O_1, O_4\}, \{A_4\}\}$
6. $\{\{O_1, O_2\}, \{A_2\}\}$
7. $\{\{O_2, O_5\}, \{A_3\}\}$
8. $\{\{O_2, O_6\}, \{A_1\}\}$
9. $\{\{O_1, O_2, O_3, O_4, O_5\}, \{\varnothing\}\}$.

The line diagram of the formal concepts generated from the context of Table 6 is shown in Fig. 4.

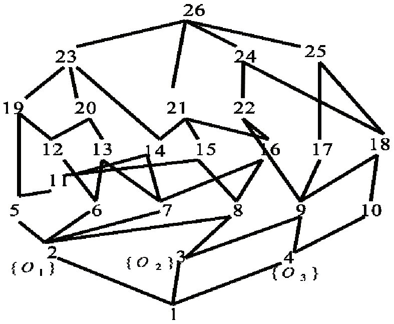

Fig. 3. Fuzzy concept lattice for the context shown in Table 5.

From Fig. 4, we can find that:

- The specialized concepts are 2, 3, and 4. Hence, objects O_1, O_2, and O_3 cover the maximal attributes of formal context, as shown in Table 6.
- The generalized concepts are 5, 6, 7 and 8. Hence, the attributes A_1, A_2, A_3 and A_4 cover the maximal objects of formal context, as shown in Table 6.

We can observe that the fuzzy concept lattice shown in Fig. 3 and its corresponding crisp lattice shown in Fig. 4 have the same specialized and generalized concepts. We can also observe that the crisp lattice structure contains a fewer number of concepts when compared to its corresponding fuzzy concept lattice. To analyze the importance of the obtained concepts, we used the metric stability in the next section.

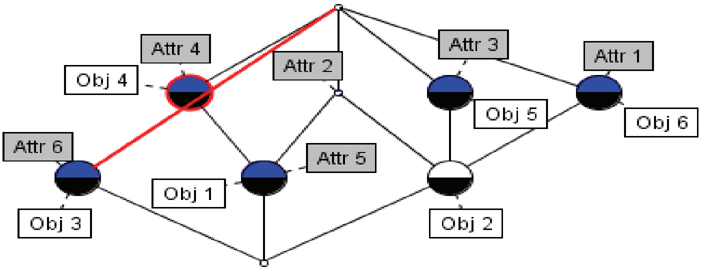

Fig. 4. Concept lattice for the context shown in Table 6.

4.2 Stability of the Obtained Formal Concepts Using the Proposed Method:

The notion of the stability of formal concepts was introduced by Kuznetsov [40]. Let **K** = (**G**, **M**, R) be a formal context and (O, A) be a formal concept of **K**. Then the stability index σ of (O, A) is defined as follows:

$$\sigma(O,A) = |\{C \subseteq O \mid C^{\uparrow} = B\}| / 2^{o},$$

This metric measures the dependency of the intent of formal concepts on the particular objects of its extent. This helps us when the given formal context changes or when some of the objects disappear. The stability indicates how likely it is for a concept to remain in the concept lattice. Stability can also be used to construct a stabilized lattice for a given threshold.

In this paper we have used this metric for formal concepts generated from the crisp context (shown in Tables 4 and 6) that was obtained using the proposed method.

The stability of formal concepts generated from Table 4 can be computed as follows:

1. $\{\{O_5\},\{A_1,A_2,A_3,A_4\}\}$
   (a) $O_5^{\uparrow} = \{A_1,A_2,A_3,A_4\}$,
   (b) $\varnothing^{\uparrow} = \{A_1,A_2,A_3,A_4\}$ by default.
   Hence, the stability of this concept is 2/2=1.
2. $\{\{O_2,O_5\},\{A_1,A_3,A_4\}\}$
   (a) $O_2^{\uparrow} = \{A_1,A_3,A_4\}$

(b) $O_5^{\uparrow} = \{A_1, A_2, A_3, A_4\}$

(c) $\{O_2, O_5\}^{\uparrow} = \{A_1, A_3, A_4\}$

(d) $\varnothing^{\uparrow} = \{A_1, A_3, A_4\}$ by default.

Hence, the stability of this concept is $3/2^2$=3/4=0.75.

3. $\{\{O_2, O_4, O_5\}, \{A_1, A_3\}\}$

(a) $O_2^{\uparrow} = \{A_1, A_3, A_4\}$

(b) $O_4^{\uparrow} = \{A_1, A_3\}$

(c) $O_5^{\uparrow} = \{A_1, A_2, A_3, A_4\}$

(d) $\{O_2, O_4\}^{\uparrow} = \{A_1, A_3\}$

(e) $\{O_2, O_5\}^{\uparrow} = \{A_1, A_3, A_4\}$

(f) $\{O_4, O_5\}^{\uparrow} = \{A_1, A_3\}$

(g) $\{O_2, O_4, O_5\}^{\uparrow} = \{A_1, A_3\}$

(h) $\varnothing^{\uparrow} = \{A_1, A_3\}$ by default.

4. Similarly, the stability of $\{\{O_1, O_2, O_3, O_4, O_5\}, \{A_1\}\}$ is 25/32=0.7.

Similarly, the stability of formal concepts generated from Table 6 can be computed as follows:

1. $\{\{\varnothing\}, \{A_1, A_2, A_3, A_4, A_5, A_6\}\}$

(a) $\varnothing^{\uparrow} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$.

Hence, the stability of this concept is 1/1=1.

2. $\{\{O_3\}, \{A_6\}\}$

(a) $O_3^{\uparrow} = \{A_6\}$

(b) $\varnothing^{\uparrow} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$.

Hence, the stability of this concept is 1/2=0.5.

3. $\{\{O_1\}, \{A_2, A_4, A_5\}\}$

(a) $O_1^{\uparrow} = A_2, A_4, A_5$

(b) $\varnothing^{\uparrow} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$.

Hence the stability of this concept is 1/2=0.5.

4. $\{\{O_2\}, \{A_1, A_2, A_3\}\}$

(a) $O_2^{\uparrow} = A_1, A_2, A_3$

(b) $\varnothing^{\uparrow} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$. Hence, the stability of this concept is 1/2=0.5.

5. $\{\{O_1, O_4\}, \{A_4\}\}$

(a) $O_1^{\uparrow} = A_2, A_4, A_5$,

(b) $O_4^{\uparrow} = A_4$,

(c) $\{O_1, O_4\}^{\uparrow} = A_4$

(d) $\varnothing^{\uparrow} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$. Hence, the stability of this concept is $2/2^2$=1/2=0.5.

6. $\{\{O_1, O_2\}, \{A_2\}\}$
   (a) $O_1^{\uparrow} = A_2, A_4, A_5$,
   (b) $O_2^{\uparrow} = A_2$,
   (b) $\{O_1, O_2\}^{\uparrow} = A_2$
   (d) $\varnothing^{\uparrow} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$. Hence, the stability of this concept is $2/2^2=1/2=0.5$.
7. $\{\{O_2, O_5\}, \{A_3\}\}$
   (a) $O_2^{\uparrow} = A_1, A_2, A_3$,
   (b) $O_5^{\uparrow} = A_3$,
   (c) $\{O_2, O_5\}^{\uparrow} = A_3$
   (d) $\varnothing^{\uparrow} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$. Hence, the stability of this concept is $2/2^2=1/2=0.5$.
8. $\{\{O_2, O_6\}, \{A_1\}\}$
   (a) $O_2^{\uparrow} = A_1, A_2, A_3$,
   (b) $O_6^{\uparrow} = A_1$,
   (c) $\{O_2, O_6\}^{\uparrow} = A_1$
   (d) $\varnothing^{\uparrow} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$. Hence, the stability of this concept is $2/2^2=1/2=0.5$.
9. Similarly, the stability of the formal concept $\{\{O_1, O_2, O_3, O_4, O_5\}, \{\varnothing\}\}$ is 0.75.

We can observe that the proposed method provides stable formal concepts at the threshold of 0.5. In the next section we provide an application for the proposed method.

Table 7. Fuzzy context by the headmaster

	A_1	A_2	A_3
O_1	0.8	0.0	0.0
O_2	0.0	0.4	0.4
O_3	0.0	0.0	0.3
O_4	0.7	0.0	0.0
O_5	0.0	0.3	0.5

5. Application of the Proposed Method

We have applied the proposed method on the fuzzy data set shown in Tables 7–9. These data sets were collected from headmasters, retired teachers, and educators, respectively, by Kandasamy and Smarandache [23] and contain 5-objects and 3-attributes. They are described as follows:

O_1 = Teaching is good,
O_2 = Teaching is poor,

O_3 = Teaching is mediocre,

O_4 = Teacher is kind (molds the character of students in the right way),

O_5 = Teacher is harsh.

The attributes are:

A_1 =Good student,

A_2 =Bad student,

A_3 =Average student.

From Table 7, Kandasamy and Smarandache [23] have concluded that:

- A harsh teacher cannot produce a good student,
- A harsh teacher can produce only average and bad students.

Table 8. Fuzzy context by retired teacher

	A_1	A_2	A_3
O_1	0.6	0.0	0.5
O_2	0.0	0.5	0.5
O_3	0.6	0.5	0.6
O_4	0.6	0.0	0.4
O_5	0.0	0.6	0.8

From Table 8 Kandasamy and Smarandache [23] have concluded that:

- A harsh teacher cannot produce good student but have equal degree to produce bad and average students.

Table 9. Fuzzy context by retired educationalist

	A_1	A_2	A_3
O_1	0.7	0.3	0.6
O_2	0.0	0.5	0.3
O_3	0.3	0.5	0.5
O_4	0.8	0.5	0.6
O_5	0.0	0.5	0.6

From Table 9, Kandasamy and Smarandache [23] have concluded that:

- A harsh teacher cannot produce good students. He/She can produce average students, but has a chance to produce bad students as well.

The final conclusions of Kandasamy and Smarandache [23] from these 3 experts are:

- A harsh teacher, due to his/her rudeness and harshness, always frightens the students. Due to this fact, he/she is certain to produce average students and also has the possibility of producing bad students.
- A harsh teacher cannot produce good students.

Table 10. Crisp order of the fuzzy context shown in Table 7

	A_1	A_2	A_3
O_1	1	0	0
O_2	0	1	1
O_3	0	0	1
O_4	1	0	0
O_5	0	1	1

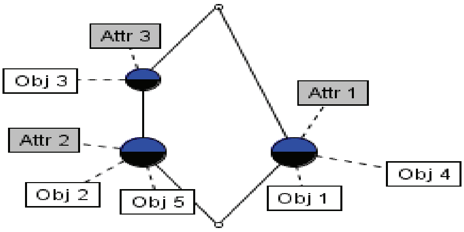

Fig. 5. Concept lattice for the context shown in Table 10.

We analyzed the fuzzy contexts shown in Tables 7–9 using the proposed method. The computed crisp contexts for these fuzzy contexts are shown in Tables 10–12, respectively.

The formal concepts generated from the context shown in Table 10 are:

1. $\{\{\varnothing\},\{A_1,A_2,A_3\}\}$
2. $\{\{O_2,O_5\},\{A_2,A_3\}\}$
3. $\{\{O_1,O_4\},\{A_1\}\}$
4. $\{\{O_2,O_3,O_5\},\{A_3\}\}$
5. $\{\{O_1,O_2,O_3,O_4,O_5\},\{\varnothing\}\}$

The lattice constructed from the formal concepts above is shown in Fig. 5 using the ConExp tool [47]. It shows the following information:

- The specialized concepts are $\{\{O_2,O_5\},\{A_2,A_3\}\}$ and $\{\{O_1,O_4\},\{A_1\}\}$.
- From the concept $\{\{O_2,O_5\},\{A_2,A_3\}\}$, we can conclude that, if the teaching is poor and the teacher is harsh, then the teacher can produce bad and average students. A harsh teacher cannot produce good students because there is no formal concept that contains object 5 and attribute 1.
- From the concept $\{\{O_1,O_4\},\{A_1\}\}$, we can conclude that, if the teaching is good and the teacher is kind, then the teacher can produce good students.

Table 11. Crisp order of the fuzzy context shown in Table 8

	A_1	A_2	A_3
O_1	1	0	0
O_2	0	1	0
O_3	1	0	1
O_4	1	0	1
O_5	0	1	1

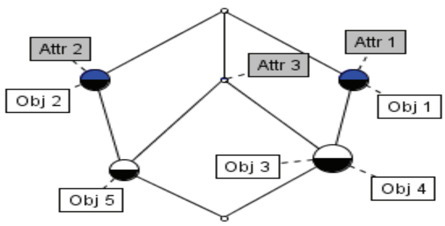

Fig. 6. Concept lattice for the context shown in Table 11.

The formal concepts generated from the context shown in Table 11 are:

1. $\{\{\varnothing\},\{A_1,A_2,A_3\}\}$,
2. $\{\{O_5\},\{A_2,A_3\}\}$,
3. $\{\{O_3,O_4,O_5\},\{A_3\}\}$
4. $\{\{O_3,O_4\},\{A_1,A_3\}\}$
5. $\{\{O_2,O_5\},\{A_2\}\}$
6. $\{\{O_1,O_3,O_4\},\{A_1\}\}$
7. $\{\{O_1,O_2,O_3,O_4,O_5\},\{\varnothing\}\}$

The lattice constructed using the above formal concepts is shown in Fig. 6 and shows the following information:

- The specialized concepts are $\{\{O_5\},\{A_2,A_3\}\}$ and $\{\{O_3,O_4\},\{A_1,A_3\}\}$.
- From the concept $\{\{O_5\},\{A_2,A_3\}\}$, we can conclude that, if the teacher is harsh, then the teacher can produce bad and average students. A harsh teacher cannot produce good students, because there is no formal concept that contains object 5 and attribute 1.
- From the concept $\{\{O_3,O_4\},\{A_1,A_3\}\}$, we can conclude that, if the teacher is mediocre and kind, then the teacher can produce good and average students.

Table 12. Crisp order of the fuzzy context shown in Table 9

	A_1	A_2	A_3
O_1	1	1	1
O_2	0	1	0
O_3	0	1	1
O_4	1	1	1
O_5	0	1	1

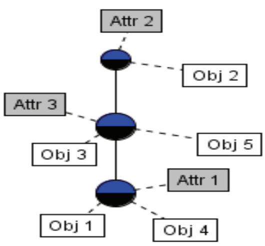

Fig. 7. Concept lattice for the context shown in Table 12.

The formal concepts generated from the context shown in Table 12 are:

1. $\{\{O_1, O_4\}, \{A_1, A_2, A_3\}\}$
2. $\{\{O_3, O_5\}, \{A_2, A_3\}\}$
3. $\{\{O_1, O_2, O_3, O_4, O_5\}, \{A_2\}\}$

The lattice obtained from the above formal concepts is shown in Fig. 7 [47]. From the concept $\{\{O_3, O_5\}, \{A_2, A_3\}\}$, we can conclude that, if a teacher is mediocre and harsh then the teacher can produce bad and average students. He/she cannot produce a good student, because there is no formal concept that contains object 5 and attribute 1.

From the lattice structures shown in Figs. 5–7 we can conclude that:

(a) A harsh teacher cannot produce good students.

(b) A harsh teacher can produce only average and bad students.

(c) If the teaching is good and the teacher is kind then he/she can produce good students.

These conclusions are similar to those obtained by Kandasamy and Smarandache [23].

6. Discussions

In this paper our aim is to reduce the number of formal concepts and the size of the lattice structure that results from a fuzzy context. Recently, some methods have been investigated [13,20,26,38,39,41-44]. Kang et al. [38] have discussed the δ-fuzzy concept lattice and the δ-rule for different granulations. Bartl et al. [41] have presented the generalization of a fuzzy relational system into a crisp relation using a composition operation by inserting hedges and its interpretation in FCA with a fuzzy setting. Horvath et al. [42] have discussed the cut of a fuzzy relation and its application in FCA with a fuzzy setting. Skowron et al. [44] have extensively studied granular computing. Kang et al. [38] discussed the crisp context at a different granulation-δ using the transitive closure of the given fuzzy formal context. The transitivity closure of a fuzzy formal context can be computed when the number of objects = number of attributes in a given fuzzy formal context (for the Case 2 of Section 3). We can observe that each of the available approaches focused on computing the crisp context at granulation using a transitive closure or composition, which takes approximately O ($|\mathbf{G}|^3$) or O ($|\mathbf{M}|^3$) complexity (based on the algorithm). The proposed method is different from all of the above approaches in the following aspects:

(1) The proposed method computes the corresponding crisp context based on the reflexivity, symmetry, and transitivity properties; and

(2) The proposed method provides the crisp context in both of the conditions when the number of objects = number of attributes or the number of objects $\neq$ number of attributes and takes O ($|\mathbf{G}| \times |\mathbf{M}|$) complexity. Furthermore, the proposed method preserves the generalization, specialization, and stability of concepts at some threshold, which increases the importance and applicability of obtained concepts.

We can observe that the proposed method provides a crisp order formal context of a given fuzzy context for both of the cases that have been mentioned. In this process, we have not focused on the uniqueness of an obtained crisp context. The concept lattice constructed from the obtained crisp context (using the proposed method) contains a fewer number of concepts while preserving the generalized and specialized concepts, as well as stability, at some threshold. This increases the applicability of the proposed method, while considering the relevant details of the underlying knowledge. Finding the fuzzy attribute implications is another problem in a given fuzzy formal context. Our proposed method provides its crisp order from which we can easily find some of the attribute implications for further analysis using the ConExp Tool [47]. This observation has a significant role in analyzing the human reasoning of relational informational systems [1-3,20-23,38,42]. Also, the proposed method can be applied in various fields like knowledge discovery and representation [1-3,14, 39,41,42-45,53,54]; information retrieval [2,3,24,45,54]; knowledge reduction [7,10,12,15,17-19,25, 29,31,32,35,39,40,48,51,52]; and association rule mining [1-3,6,16,46,49,50].

7. Conclusion

In this paper we aimed at providing a method for knowledge reduction in a fuzzy formal context by introducing a crisp order relation. The proposed method reduces the number of concepts and the size of the lattice structure obtained from a fuzzy context. We have also shown the step-by-step illustrations of the proposed method.

The outline of the study is as follows:

- The proposed method computes the crisp order relation for the given fuzzy relation using the properties of reflexivity, symmetry, and transitivity.
- The corresponding crisp order relation provides a fewer number of concepts that have been obtained from a fuzzy context and further reduces the size of the lattice structure.
- While reducing the number of concepts, the proposed method preserves the specialized and generalized concepts and their stability with some thresholds.

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